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ORIGINAL PAPER

Proposed connection between critical exponents and fractal dimensions in the Ising model

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Abstract Scaling properties near the critical point indicates the existence of self-similarity behavior for the critical phenomena. Although the system considered here is not a truly dynamic one, we propose a specific set of relations between fractal dimensions and critical exponents in the Ising model of statistical mechanics. In particular, we put forward, corresponding to six critical exponents for the Ising model, six fractal dimensions. Assuming the latter proposals, we can then derive relationships between such fractal dimensions.

Keywords Critical exponents · Fractal dimensions · The Ising model · Critical phenomena

1 Introduction

The critical phenomena exist in various second-order phase transitions, which have attracted extensive interest [1,2]. On the other route, the dynamic complexity has been intensively investigated in various areas [3–5]. The definition of fractals has been shown to be applicable for a wide variety of disciplines: such as physics, chemistry,

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Table 1 Critical exponents α, β, γ, δ, η and ν of Ising systems

Ising	α	β	γ	δ	η	ν
1D	–	–	2	∞	1	2
2D	0	$\frac{1}{8}$	$\frac{7}{4}$	15	$\frac{1}{4}$	1
3D	0	$\frac{3}{8}$	$\frac{5}{4}$	$\frac{13}{3}$	$\frac{1}{8}$	$\frac{2}{3}$
4D	0	$\frac{1}{2}$	1	3	0	$\frac{1}{2}$

The exact values for dimensionalities 1, 2 and 4 of the Ising model are listed together with those obtained by means of two conjectures for three dimensions in [6]

biology, neurology, astronomy, geophysics, meteorology and economics. The scaling properties near the critical point indicate the existence of the self-similarity behavior for the critical phenomena. Although the Ising system considered here is not a truly dynamic one, analogous to the definition of the fractal dimensions in other systems, it seems reasonable for us to consider that physical quantities near a critical point could be measured by some parameters (i.e., temperature and a magnetic field) so that the fractal dimensions could be defined also for the critical phenomena.

One of us (ZDZ [6]; see also [7] and [8]) has, by means of two conjectures, obtained precise proposals for the six critical exponents of the Ising model Hamiltonian in three dimensions (d = 3). Using previously known results for d = 1, 2 and 4, Table 1 has thereby been constructed. Our purpose below is to propose definitions of the fractal dimensions for the critical behaviour summarized in Table 1. However, before we turn to these definitions, let us summarize well-established connections between the critical exponents already displayed in Table 1 (see also Klein and March [9] for a brief summary of the four relations we appeal to). The first goes back at least to Rushbrooke [10] and reads

$$\alpha + 2\beta + \gamma = 2 \tag{1}$$

while Griffiths [11] proposed the result

$$\gamma = \beta(\delta - 1). \tag{2}$$

Thirdly we have the result of Fisher [12] that

$$\gamma = \nu(2 - \eta). \tag{3}$$

But the most crucial result in the context of the present article is due to Josephson [13] which, unlike Eqs. (1)–(3), involves directly the dimensionality and reads

$$\nu d = 2 - \alpha. \tag{4}$$

With the above as background to the critical behaviour of the Ising model in d dimensions, we turn in Sect. 2 immediately below to propose definitions of the fractal dimension d_f associated with the six critical exponents entering the relations (1)–(4).

2 Proposed definitions of the fractal dimensions for critical phenomena

According to the definition of the fractal dimensions, and considering the real dimensions of the system, we define fractal dimensions associated with the six critical exponents entering the relations (1)–(4) as follows:

$$d_f^{\beta} = d - \beta \tag{5}$$

$$d_f^{\delta} = d - \frac{1}{\delta} \tag{6}$$

$$d_f^{\alpha} = d + \alpha \tag{7}$$

$$d_f^{\gamma} = d + \gamma \tag{8}$$

$$d_f^{\nu} = d + \nu \tag{9}$$

and finally

$$d_f^{\eta} = 2d - 2 + \eta. \tag{10}$$

We next demonstrate the relations between these definitions of the fractal dimensions for critical phenomena.

2.1 Relations between proposed definitions (5)–(10) for fractal dimensions for critical properties

From a particular linear combination of Eqs. (5), (7), and (8), it may readily be verified that

$$d_f^{\alpha} - 2d_f^{\beta} + d_f^{\gamma} = \alpha + 2\beta + \gamma = 2. \tag{11}$$

where the last step in reading Eq. (11) have invoked the Rushbrooke equality (1).

Secondly, by forming $d - d_f^{\alpha}$ from Eq. (7), which immediately is seen to equal minus α, and $d_f^{\nu} - d = \nu$ from Eq. (9), we see that the relation

$$2 - d_f^{\alpha} + d = d\left(d_f^{\nu} - d\right) \tag{12}$$

is equivalent to Josephson relation (4). Thirdly, using again the result given above, for $d_f^{\nu} - d (= \nu)$ and that from Eq. (8) $d_f^{\gamma} - d = \gamma$, it follows that

$$\frac{d_f^{\gamma} - d}{d_f^{\nu} - d} = \frac{\gamma}{\nu},$$

Table 2 Critical dimensions of Ising systems

Ising	d_f^α	d_f^β	d_f^γ	d_f^δ	d_f^η	d_f^ν
1D	1	1	3	1	1	3
2D	2	$\frac{15}{8}$	$\frac{15}{4}$	$\frac{29}{15}$	$\frac{9}{4}$	3
3D	3	$\frac{21}{8}$	$\frac{17}{4}$	$\frac{36}{13}$	$\frac{33}{8}$	$\frac{11}{3}$
4D	4	$\frac{7}{2}$	5	$\frac{11}{3}$	6	$\frac{9}{2}$

All the values for the fractal dimensions are derived from the corresponding critical exponents listed in Table 1 by using Eqs. (5)–(10)

But from Eq. (10) $2d - d_f^\eta = 2 - \eta$ which by appealing to the Fisher result (3) is equal to γ/ν. Hence we reach the relation

$$\left(d_f^\nu - d\right)\left(2d - d_f^\eta\right) = d_f^\gamma - d. \tag{13}$$

Finally, but without going into details, we record the fourth such relation as

$$\left(d - d_f^\delta\right)\left(d_f^\gamma - d\right) = \left(d - d_f^\beta\right)\left(1 - d + d_f^\delta\right). \tag{14}$$

This is verified by detailed use of Eqs. (2), (5), (6) and (8).

2.2 Explicit forms proposed for critical dimensions of Ising systems

To complete this section, we utilize the Ising critical exponents already collected in Table 1, plus the proposed definitions of fractal dimensions in Eqs. (5)–(10), to construct Table 2. As anticipated in the title and abstract, twelve out of eighteen entries have fractal character if we focus on dimensionalities 2, 3 and 4 of the Ising model.

It could be concluded from Table 2 that for the 1D Ising model, the fractal dimensions d_f^α, d_f^β, d_f^δ and d_f^η are the same as the spatial dimensionality d. The values for d_f^α are equal to the real dimensions of the system for d = 1, 2, 3, and 4, revealing that measuring the specific heat C_V is very particular, which shows logarithmic singularity at the critical point (for d = 2 and 3). The fractal dimension d_f^β or d_f^δ for measuring the order parameter (i.e., magnetization M) is smaller than the dimensionality d. This is consistent with the fact that the blocks of the correlated spins near the critical point do not fill the whole space of the system. On the contrary, other fractal dimensions are larger than the dimensionality d. It means that measuring other physical properties, like the magnetic susceptibility χ, the correlation length ξ and the spin–spin correlation function $G^{(2)}(r)$, differs with that for the magnetization. It is analogous to measure those properties in the space with the dimensionality higher than the spatial dimensions of the Ising system. For the Ising model, the fractal dimension for measuring a fixed property (except the specific heat C_V) increases with increasing the dimensionality d.

3 Summary and future directions

The main focal points of the present article are (i) the definitions proposed of the fractal dimensions for critical properties defined in Eqs. (5)–(10) and (ii) the specific values thereby obtained and shown in Table 2 for Ising systems with dimensionalities 2, 3 and 4 inclusive.

For the future, it will of course be of considerable interest if it proves possible to connect our predictions with, say, practical magnetism, and if one understands deeper physical insights of these fractal dimensions. A further direction of potential interest for the future lies in quantum mechanics and, to be quite specific in one-body potential theory of atoms [14]. We conclude by referring briefly to the use of the fractional Schrödinger equation (FSE) [15] to treat such fractal dimensionality in atoms. If the one-body potential energy is denoted by $V(\mathbf{r})$, which has an additive contribution, as yet unknown, from exchange (x) and correlation (c), namely $V_{xc}(\mathbf{r})$, then one could use the FSE to calculate one-body eigenfunctions $\psi_i(\mathbf{r})$ and corresponding eigenvalues $\in_i$. We only note in the semiclassical Thomas-Fermi (TF) limit [15] that the so-called Slater sum $S(\mathbf{r}, \beta)$ is defined exactly by

$$S(r,\beta) = \sum_{alli} \psi_i(r)\psi_i * (r)\exp(-\beta \in_i), \tag{15}$$

becomes, in the semiclassical TF approximation [16]; see also [17]

$$S_{\mathrm{TF}}(r,\beta) = \frac{1}{(2\pi\beta)^{d/2}} \exp\left(-\beta V(r)\right) \tag{16}$$

where $\beta = 1/k_B T$, with k_B denoting Boltzmann's constant. It seems natural enough, at least in semiclassical theory, to use Eq. (16), which is demonstrably semiclassically correct for all integral d, for fractal dimension. For the moment, however, transcending the TF approximation (16) is hampered by the shortage of exact analysis of the FSE for even relatively simple one-body potential $V(\mathbf{r})$. But more analytical progress can be expected in this quantum mechanical area in the foreseeable future.

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